Math 245C Lecture 1 Notes

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1 Functions of Bounded Variation and Distribution Functions

1.1 Functions of bounded variation

First, let's review the idea of functions of bounded variation.

Definition 1.1. Let $-\infty < a < b < \infty$. We say that $f : [a, b] \to \mathbb{R}$ is of **bounded variation** and write $f \in \mathrm{BV}([a, b])$ if

$$\sup_n \sup_{x_i} \left\{ \sum_{i=1}^{n-1} |f(x_i) - f(x_{i-1})| : a = x_0 < x_1 < \cdots < x_n = b \right\} < \infty$$

We call this supremum the **total variation norm** and write it as $\|f\|_{\mathrm{TV}([a,b])}$.

If $f : [a, b] \to \mathbb{R}$, we write $f' = f'_{\mathrm{abs}} + f'_{\mathrm{sing}}$, where $\int |f'_{\mathrm{abs}}| + \int |f'_{\mathrm{sing}}| < \infty$.

Definition 1.2. We sat that $F : \mathbb{R} \to \mathbb{C}$ is of bounded variation if

$$\sup_{x_0, x_1} \left\{ \|F\|_{\mathrm{TV}([x_0,x_1])} : -\infty < x_0 < x_1 < \infty \right\} < \infty.$$

Set $T_F(x) = \sup_{x_0 < x} \|F\|_{\mathrm{TV}([x_0,x])}$. This is a monotone increasing function. Observe that $F \in \mathrm{BV}(\mathbb{R})$ means that $\lim_{x\to\infty} T_F(x) < \infty$.

We can normalize functions of bounded variation.

Definition 1.3. $\mathrm{NBV}(\mathbb{R})$ is the set of $F \in \mathrm{BV}(\mathbb{R})$ such that

1. F is right continuous.
2. $\lim_{x\to-\infty} F(x) = 0$.

Definition 1.4. If ν_1, ν_2 are two signed Borel measures on $\mathbb{R}$ of finite total mass, $\nu = \nu_1 + i\nu_2$ is called a **complex Borel measure**.

Remark 1.1. Signed measures can take the values $\pm\infty$, but we require them to be finite here.

Proposition 1.1. *If $F \in \mathrm{NBV}(\mathbb{R})$, then there exists a unique Borel complex measure μ_F on $\mathbb{R}$ such that $F(x) = \mu_F((-\infty, x])$. Conversely, every Borel complex measure is of the form μ_F.*

Theorem 1.1 (integration by parts)**.** *Let $F, G \in \mathrm{BV}([a,b])$, where $-\infty < a < b < \infty$. Assume F is right continuous and G is continuous. Then*

$$\int_{(a,b]} F(x)\, d\mu_G(x) + \int_{(a,b]} G(x)\, d\mu_F(x) = F(b)G(b) - F(a)G(a).$$

Remark 1.2. One uses the notation

$$\int_{(a,b]} F(x)\mu_G(x) = \int_{(a,b]} F(x)\, dG(x).$$

1.2 Distribution functions

Throughout this section, $(X, \mathcal{M}, \mu)$ is a measure space, and $0 < p < \infty$.

Definition 1.5.

$$L^p(X, \mu) = \left\{ F : X \to \mathbb{C} : F \text{ is measurable}, \int_X |F|^p\, d\mu < \infty \right\}.$$

We write

$$\|F\|_{L^p} = \left(\int_X |F(x)|^p\, d\mu(x) \right)^{1/p}.$$

Remark 1.3. We will write L^p or $L^p(\mu)$ for $L^p(X, \mu)$.

Proposition 1.2 (Chebyshev's inequality)**.** *Fix $\alpha > 0$.*

$$\int_X |F(x)|^p\, d\mu(x) \geq \alpha^p \mu(\{|F| > \alpha\}).$$

Proof.

$$\int_X |F(x)|^p\, d\mu(x) \geq \int_{\{|F|>\alpha\}} |F(x)|^p\, d\mu(x) \geq \int_{\{|F|\geq\alpha\}} \alpha^p\, d\mu(x) \geq \alpha^p \mu(\{|F| > \alpha\}). \qquad \square$$

Remark 1.4. If $F \in L^p$, then

$$\sup_{\alpha>0} \alpha^p \mu(\{|F| > \alpha\}) \leq \|F\|_{L^p}^p < \infty.$$

Definition 1.6. Let $F : X \to \mathbb{C}$ be measurable. The **distribution function** of F is $\lambda_F : (0, \infty) \to [0, \infty]$ defined as $\lambda_F(\alpha) = \mu(\{|F| > \alpha\})$.

Proposition 1.3. *Let $F, G :\to \mathbb{C}$ be measurable.*

1. *λ_F is monotone decreasing.*
2. *If $|F| \leq |G|$, then $\lambda_F \leq \lambda_G$.*
3. *If $H := F + G$, then $\lambda_H(\alpha) \leq \lambda_F(\alpha/2) + \lambda_G(\alpha/2)$.*
4. *If $F_n : X \to \mathbb{C}$ are measurable functions such that $|F_n| \leq |F_{n+1}| \leq |F|$ for all n, and $\lim_n |F_n| = |F|$, then $\lim_n \lambda_{F_n} = \lambda_F$.*

Proof. Define $E(\alpha, F) = \{|F| > \alpha\}$ for $\alpha > 0$.

1. If $0 < \alpha_1 < \alpha_2$, then $E(\alpha_2, F) \subseteq E(\alpha_1, F)$. So
$$\lambda_F(\alpha_2) = \mu(E(\alpha_2, F)) \leq \mu(E(\alpha_1, F)) = \lambda_F(\alpha_1).$$
2. If $|F| \leq |G|$, then for $\alpha > 0$, $E(\alpha, F) \subseteq E(\alpha, G)$.
3. If $|H| > \alpha$, then $|F| + |G| \geq |F + G| = |H| > \alpha$. Then $|F| > \alpha/2$ or $|G| > \alpha/2$. So $E(\alpha, H) \subseteq E(\alpha/2, F) \cup E(\alpha/2, G)$. So
$$\mu(E(\alpha, H)) \leq \mu(E(\alpha/2, F)) + \mu(E(\alpha/2, G)).$$
4. Let $(F_n)_n$ be as above. Then $\lambda_{F_n} \leq \lambda_{F_{n+1}} \leq \lambda_F$. Hence, $\lim_n \lambda_{F_n}$ exists and is $\leq \lambda_F$. To get the reverse inequality, we use
$$E(\alpha, F) = \bigcup_{n=1}^{\infty} E(\alpha, F_n).$$
To get the $\subseteq$ containment, if $|F(x)| > \alpha$, then there exists n such that $|F_n(x)| > \alpha$. Note that $E(\alpha, F_n) \subseteq E(\alpha, F_{n+1}) \subseteq E(\alpha, F)$ for all n. Since μ is a measure,
$$\mu(E(\alpha, F)) = \mu\left(\bigcup_{n=1}^{\infty} E(\alpha, F_n)\right) = \lim_n \mu(E(\alpha, F_n)).$$ □

Definition 1.7. Weak L^p, denoted $L^p(\mu, \text{weak})$, os the set of measurable functions $F : X \to \mathbb{C}$ such that $[F]_p < \infty$, where
$$[F]_p = \sup_{\alpha \in (0,\infty)} \alpha^p \lambda_F(\alpha).$$

Remark 1.5. $L^p(\mu) \subseteq L^p(\mu, \text{weak})$.

These are not the same. What is the difference? We will show that being in weak L^p is equivalent to $\int_0^\infty \alpha^{p-1} \lambda_F(\alpha)\, d\alpha < \infty$. So $F \in L^p$ means that $\alpha^{p-1}\lambda_F \in L^1((0, \infty))$, while $F \in L^p(\mu, \text{weak})$ means that $\alpha^p \lambda_F \in L^\infty(0, \infty)$.